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Symplectic geometry and isomonodromic deformations.
Singular flat $Gl_n(\mathbb{C})$ connections over $\mathbb{C}P^1$ are viewed in three ways, giving different approaches to their moduli: 1) as spaces of coefficients of linear meromorphic differential systems on a trivial holomorphic vector bundle, 2) using an infinite dimensional C^∞ viewpoint, and 3) via Stokes matrices (which are the analogue of monodromy matrices for such connections). Intrinsic symplectic structures are exhibited on the spaces in 1) and 2) and the natural map between them is formally shown to be symplectic. These results yield a symplectic description of the equations for isomonodromic deformations (in the sense of Jimbo-Miwa-Ueno) which will also be presented.

Symplectic Geometry and Isomonodromic Deformations

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Themes include:

- Flows on Complex Coadjoint Orbits
- Moduli of Meromorphic Connections
- Integrable Nonlinear Differential Equations
- Families of ACIHSs

Introduction

The perspective here is that the differential equations for isomonodromic deformations (IMDs) arise naturally from the geometry of moduli spaces of meromorphic connections on the Riemann sphere, $\mathbb{P}^1(\mathbb{C})$.

These equations enjoyed a systematic study by Jimbo-Miwa-Ueno [18] where they were first derived and their complete integrability proven. The Painlevé property of these equations was demonstrated later by Miwa [21].

This work continues the study of isomonodromic deformations, bringing out the symplectic geometry of the equations in a coordinate free fashion. A symplectic approach to the simplest IMD equations has been given by a number of people using somewhat specialised methods (see Okamoto [22] and Harnad-Wisse [12] for the Painlevé equations and Hitchin [13] for the Schlesinger equations). Although they are the simplest IMD equations, the Painlevé and Schlesinger equations are far from simple; they occur in many deep problems. The main result presented here is a unified description of the symplectic nature of the full family of IMD equations studied by Jimbo-Miwa-Ueno. The IMD equations are a flat symplectic (Ehresmann) connection on a bundle of symplectic manifolds over the space of deformation parameters. A new feature arising in the general case is that this bundle of symplectic manifolds is not a priori trivial.

This result is obtained from the complex symplectic geometry of the moduli space of meromorphic connections. Symplectic aspects of such moduli spaces have been studied previously only in the case where the connections have simple poles (which is closely related to spaces of flat connections on the punctured surface). Thus a large part of this work is devoted to finding/describing a natural symplectic structure in the arbitrary order pole case.

The applications of isomonodromic deformations to mathematical and physical problems will not be discussed although it should be noted that IMDs underlie most (perhaps all) integrable nonlinear partial differential equations. For example the KdV equation has a reduction to the first Painlevé equation. Also, geometrically the IMD equations are closely analogous to the Gauss-Manin connection in nonabelian cohomology.

The original impetus for this work came from B.Dubrovin's work on Frobenius manifolds [9] together with N.Hitchin's distillation [13]. The idea that a perspective such as that outlined here might hold for arbitrary isomonodromic deformations germinated after the 1996 Isomonodromy Conference in Luminy. The lectures of H.Flaschka on the work [15, 16] of K.Iwasaki were particularly relevant. In a sense the work here is complementary to Iwasaki's: this work studies deformations of meromorphic connections with arbitrary order poles over $\mathbb{P}^1$ whereas Iwasaki studies Fuchsian equations ($\sim$ connections with simple poles) over arbitrary compact Riemann surfaces.

This work will appear in the author's Oxford D.Phil thesis. He would like to thank the many people who have helped him along the way, especially his supervisor Nigel Hitchin.

Summary

This work may be summarised in the following five points:

1) Polar Parts Manifolds $\mathcal{P}(\mathbf{A}), \mathcal{P}_{\text{ext}}(\mathbf{A})$

• The set of isomorphism classes of 'nice' meromorphic connections on a trivial holomorphic vector bundle over $\mathbb{P}^1(\mathbb{C})$ is described explicitly and is shown to have a natural complex Poisson structure.

• The symplectic leaves are obtained by fixing the 'formal equivalence classes' at the poles and are referred to as the 'Polar parts manifolds' $\mathcal{P}(\mathbf{A})$ (where $\mathbf{A}$ encodes the choice of formal equivalence classes). They are described in terms of complex coadjoint orbits.

• $\mathcal{P}(\mathbf{A})$ is identified with the algebraically completely integrable Hamiltonian systems (polynomial matrix systems) of Beauville [7] and Adams-Harnad-Hurtubise [1].

• An 'extended' version $\mathcal{P}_{\text{ext}}(\mathbf{A})$ is defined by incorporating 'compatible framings' and not fixing the residues of the formal equivalence classes. $\mathcal{P}_{\text{ext}}(\mathbf{A})$ is a genuine symplectic manifold ($\mathcal{P}(\mathbf{A})$ may in fact be non-Hausdorff). $\mathcal{P}(\mathbf{A})$ is a finite dimensional symplectic quotient of $\mathcal{P}_{\text{ext}}(\mathbf{A})$.

2) Monodromy Manifolds $M(\mathbf{A}), M_{\text{ext}}(\mathbf{A})$

• A 'generalised monodromy manifold' $M(\mathbf{A})$ is defined explicitly (following Jimbo-Miwa-Ueno [18]) in terms of 'Stokes matrices' and 'connection matrices'.

• $M(\mathbf{A})$ describes the set of isomorphism classes of meromorphic connections with fixed formal equivalence classes on *arbitrary* degree zero holomorphic vector bundles over $\mathbb{P}^1$.

• Again an 'extended' version $M_{\text{ext}}(\mathbf{A})$ is defined by incorporating compatible framings and allowing free diagonal residues. It is a complex manifold having $M(\mathbf{A})$ as a subquotient.

3) $\mathbf{C}^\infty$ Approach to Meromorphic Connections

• By studying flat C^∞ singular connections ('C^∞ connections with poles') the notion of formal equivalence is captured in a C^∞ way. Fixing the formal equivalence classes corresponds to fixing the 'Laurent expansions' of the C^∞ singular connections.

• The monodromy manifolds $M(\mathbf{A})$ and $M_{\text{ext}}(\mathbf{A})$ are realised in terms of gauge equivalence classes of flat singular connections.

• This description yields (at least formally) a symplectic structure on the monodromy manifolds $M(\mathbf{A})$ and $M_{\text{ext}}(\mathbf{A})$ following the approach of Atiyah-Bott [3].

4) The Monodromy Map is Symplectic

By thinking in terms of meromorphic connections there is evidently an injective map $\nu : \mathcal{P}_{\text{ext}}(\mathbf{A}) \to M_{\text{ext}}(\mathbf{A})$ from the extended polar parts manifold to the extended monodromy manifold. This will be called the (generalised) monodromy map and can be thought of as a generalised Riemann-Hilbert morphism. It is known to be complex analytic and that $\dim(\mathcal{P}_{\text{ext}}(\mathbf{A})) = \dim(M_{\text{ext}}(\mathbf{A}))$. The key result here is:

• The monodromy map $\nu : \mathcal{P}_{\text{ext}}(\mathbf{A}) \to M_{\text{ext}}(\mathbf{A})$ is symplectic.

5) Isomonodromic Deformations (See Figure 1)

• The extended polar parts and monodromy manifolds form fibre bundles $\mathcal{P}$ and M respectively over a space X of deformation parameters (encoding the positions of the poles and the the irregular parts of the formal equivalence classes).

• The monodromy bundle M has a flat Ehresmann connection on it, transverse to the fibres of the map to X (essentially given by keeping the Stokes and connection matrices constant). This identifies the monodromy data at different values of the deformation parameters.

By pulling back this connection along the monodromy map, a flat connection is induced on the polar parts bundle $\mathcal{P}$. The horizontal leaves of this connection on $\mathcal{P}$ correspond to meromorphic connections on $\mathbb{P}^1$ with the *same* generalised monodromy; the isomonodromic deformation equations are precisely the equations determining these horizontal leaves. The main result is then:

• The pulled back connection on the polar parts bundle $\mathcal{P}$ is symplectic. That is, the local analytic diffeomorphisms induced by the IMD equations between different fibres (extended polar parts manifolds) are symplectomorphisms.

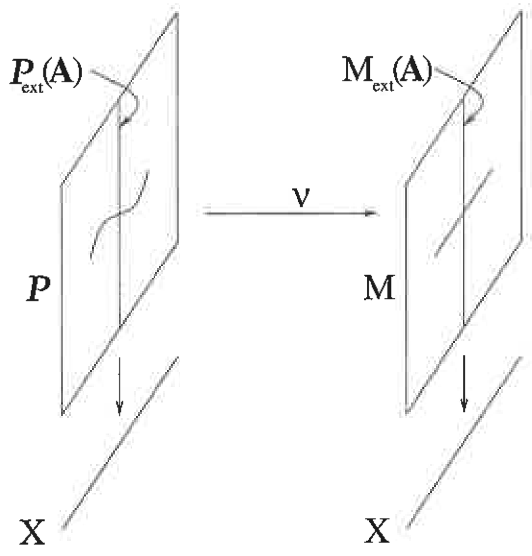

FIGURE 1. Isomonodromic Deformations

1. MORE DETAILS

The rest of this poster gives the definitions and provides more details about the statements in the summary.

Choose m distinct points $a_1, \dots, a_m \in \mathbb{P}^1$ and *negative* integers $k_1, \dots, k_m$. Let $D = -k_1(a_1) - \cdots - k_m(a_m)$ be the corresponding effective divisor. For simplicity choose a coordinate z on $\mathbb{P}^1$ with respect to which each a_i is finite.

Firstly we will study spaces of pairs (V, ∇) where V is a degree zero rank n holomorphic vector bundle over $\mathbb{P}^1$ and ∇ is a meromorphic connection on V:

$$\nabla : V \to V \otimes K(D)$$

satisfying the usual Leibniz rule, where K is the canonical sheaf on $\mathbb{P}^1$. Concretely in a local trivialisation at a_i, ∇ has the Laurent series:[1]

$$\nabla = d + {}^iA_{k_i}(z-a_i)^{k_i}dz + {}^iA_{k_i+1}(z-a_i)^{k_i+1}dz + \cdots$$

for $n \times n$ matrices ${}^iA_j$ $(j \geq k_i)$. Four important preliminary definitions are given in

Definition 1.1.

• A meromorphic connection ∇ is 'nice' if in a local trivialisation at each a_i the leading coefficient ${}^iA_{k_i}$ is

1) diagonalisable with distinct eigenvalues and $k_i \leq -2$, or

2) diagonalisable with no eigenvalues differing by nonzero integers and $k_i = -1$.

This condition is independent of the trivialisation and coordinate choice.

• Two meromorphic connections are 'formally equivalent at a_i' if there is a formal gauge transformation relating their Laurent series at a_i. Explicitly in a local trivialisation over a_i, if $\nabla' = d + {}^iB_{k_i}(z-a_i)^{k_i}dz + \cdots$, then ∇ and ∇' are formally equivalent at a_i if there exists $F \in GL_n(\mathbb{C}[\![(z-a_i)]\!])$ such that as power series:

$$F({}^iA_{k_i}(z-a_i)^{k_i}dz + \cdots)F^{-1} - F^{-1}dF = {}^iB_{k_i}(z-a_i)^{k_i}dz + \cdots$$

• A 'formal normal form at a_i' is a diagonal connection germ with no holomorphic part:

$$d + {}^iA^0 = d + {}^iA^0{}_{k_i}\frac{dz}{(z-a_i)^{|k_i|}} + \cdots + {}^iA^0{}_{-1}\frac{dz}{(z-a_i)}$$

where each ${}^iA^0{}_j$ is diagonal. It is well known that within each nice formal equivalence class there is a formal normal form. The 'irregular part' of the formal normal form ${}^iA^0$ is the part with poles of order at least 2: ${}^iA^0{}_{k_i}(z-a_i)^{k_i}dz + \cdots + {}^iA^0{}_{-2}dz/(z-a_i)^2$.

• A 'compatible framing at a_i' of a nice connection ∇ on a vector bundle V is a choice of isomorphism between the fibre V_{a_i} and $\mathbb{C}^n$, ${}^ig : V_{a_i} \to \mathbb{C}^n$, such that the leading term ${}^iA_{k_i}$ of ∇ is *diagonal* in any local trivialisation of V extending ${}^ig$.

From now on we will restrict to nice connections since these are simplest yet sufficient for our purposes (to describe the symplectic nature of isomonodromic deformations). In fact, to simplify the presentation here it will further be assumed that at any simple pole

[1] Pre-superscripts ${}^iA$, when used, will denote local information near $a_i \in \mathbb{P}^1$.

(where $k_i = -1$) the residue ${}^iA^0{}_{-1}$ has distinct eigenvalues, although most of the results will hold with minor modification for arbitrary nice connections.

2. POLAR PARTS MANIFOLDS

Choose a nice formal normal form ${}^iA^0$ at each a_i and let $\mathbf{A}$ denote this m-tuple of formal normal forms. Let $\mathcal{P}(\mathbf{A})$ denote the set of isomorphism classes of pairs (V, ∇) where V is a *trivial* rank n holomorphic vector bundle over $\mathbb{P}^1$ and ∇ is a meromorphic connection on V with formal normal form ${}^iA^0$ at a_i for each i and no other poles.

The main result of this section, Theorem 2.5, is the description of this set as a complex symplectic quotient of a product of complex coadjoint orbits by $GL_n(\mathbb{C})$. As usual for quotients of affine varieties by reductive groups, $\mathcal{P}(\mathbf{A})$ may not be Hausdorff but will have a dense open subset that is a genuine complex symplectic manifold. Moreover the symplectic structure obtained in this way is intrinsic (that is, independent of the coordinate choice made in order to obtain this description).

A similar description is given of the extended polar parts manifolds $\mathcal{P}_{\text{ext}}(\mathbf{A})$ describing the set of isomorphism classes of triples $(V, \nabla, \mathbf{g})$ consisting of a nice meromorphic connection ∇ on trivial V having fixed irregular part of formal normal form at each a_i (but with arbitrary residue of formal normal form) along with an m-tuple of compatible framings $\mathbf{g} = ({}^1g, \ldots, {}^mg)$ (one at each a_i). These extended manifolds seem to be the most natural level at which to study isomonodromic deformations. They are (genuine) complex symplectic manifolds, they have the polar parts manifolds as symplectic quotients and have the intriguing property that their symplectomorphism class is not dependent on the choice $\mathbf{A}$ of formal normal forms. The idea of using extended spaces has roots in the original work of Jimbo-Miwa-Ueno [18] and in work of L.Jeffrey and J.Huebschmann (see [17, 11]).

2.1. **Poisson Structure.** To start with, if ∇ is a meromorphic connection on a holomorphically trivial vector bundle $V \to \mathbb{P}^1$ with poles of order $|k_i|$ at the a_i's, then in any trivialisation of V it is of the form:

$$\nabla = d + \sum_{i=1}^{m} {}^iA_{k_i} \frac{dz}{(z-a_i)^{|k_i|}} + \cdots + {}^iA_{-1} \frac{dz}{(z-a_i)} \tag{1}$$

for some $n \times n$ matrices ${}^iA_j$ $(k_i \le j \le -1)$.

Definition 2.1.

- The 'polar part of ∇ at a_i' is ${}^iA_{k_i} \frac{dz}{(z-a_i)^{|k_i|}} + \cdots + {}^iA_{-1} \frac{dz}{(z-a_i)}$.
- The collection of tuples of $n \times n$ matrices ${}^iA_j$ $(k_i \le j \le -1)$ will be denoted by W:

$$W \cong \text{End}(\mathbb{C}^n)^{|k_1| + \cdots + |k_m|}$$

Firstly observe that the group $GL_n(\mathbb{C})$ acts by conjugation on W and two connections are isomorphic if and only if they map to the same $GL_n(\mathbb{C})$ orbit in W. Secondly, a collection of matrices $\{{}^iA_j\} \in W$ determines a meromorphic connection via the expression

(1) with poles only at the a_i's if and only if there is no further pole at ∞, which is equivalent to the sum of the residues being zero:

$$ {}^1A_{-1} + \cdots + {}^mA_{-1} = 0 \tag{2} $$

These two facts fit together naturally in Poisson geometry: somewhat remarkably, the left-hand side of (2) (the sum of the residues) can be interpreted as the moment map for the action of $GL_n(\mathbb{C})$ on W, once a suitable Poisson structure is defined. This Poisson structure on W is defined as follows. For a negative integer k define the complex Lie group:

$$ G_k := GL_n(\mathbb{C}[\zeta]/(\zeta^{|k|})) = \{g_0 + g_1\zeta + \cdots + g_{|k|-1}\zeta^{|k|-1} \big| \det(g_0) \neq 0\} $$

The corresponding Lie algebra $\mathfrak{g}_k$ has elements of the form $X = X_0 + X_1\zeta + \cdots + X_{|k|-1}\zeta^{|k|-1}$ for $n \times n$ matrices X_i. Elements of the dual $\mathfrak{g}_k^*$ of the Lie algebra are written suggestively as

$$ A = A_k \frac{d\zeta}{\zeta^{|k|}} + \cdots + A_{-1}\frac{d\zeta}{\zeta} \tag{3} $$

The pairing between $\mathfrak{g}_k^*$ and $\mathfrak{g}_k$ is given by

$$ \langle A, X \rangle = \mathrm{Res}_0(\mathrm{Tr}(AX)) = \sum_{i=k}^{-1} \mathrm{Tr}(A_i X_{-1-i}) $$

where $\mathrm{Res}_0 : \mathfrak{g}_k^* \to \mathrm{End}(E)$ is the residue map, picking out the coefficient of $d\zeta/\zeta$.

Since it is the dual of a Lie algebra, $\mathfrak{g}_k^*$ has a natural (Kostant-Kirillov) Poisson structure. Thus a Poisson structure is obtained on W from the isomorphism

$$ W \cong \mathfrak{g}_{k_1}^* \times \cdots \times \mathfrak{g}_{k_m}^*; $$

$$ \{{}^iA_j\} \mapsto \left(({}^1A_{k_1}\zeta^{k_1} + \cdots + {}^1A_{-1}\zeta^{-1})d\zeta, \ldots, ({}^mA_{k_m}\zeta^{k_m} + \cdots + {}^mA_{-1}\zeta^{-1})d\zeta\right) $$

implicit in (1) and (3). It is then true that the 'sum of the residues' is a moment map for the $GL_n(\mathbb{C})$ action on W:

$$ \mu : W \to \mathrm{End}(\mathbb{C}^n); \quad \{{}^iA_j\} \mapsto {}^1A_{-1} + \cdots + {}^mA_{-1} $$

Putting all this together gives:

Proposition 2.2. *The set of isomorphism classes of pairs (V, ∇) where V is a trivial rank n holomorphic vector bundle over $\mathbb{P}^1$ and ∇ is a meromorphic connection on V with poles of order at most $|k_i|$ at a_i is isomorphic to*

$$ \mathfrak{g}_{k_1}^* \times \cdots \times \mathfrak{g}_{k_m}^* /\!/ GL_n(\mathbb{C}) = \mu^{-1}(0)/GL_n(\mathbb{C}) $$

and so inherits a Poisson structure.

Remark 2.3. This situation has been previously studied by Beauville [7] and Adams-Harnad-Hurtubise [1] who prove that $\mathfrak{g}_{k_1}^* \times \cdots \times \mathfrak{g}_{k_m}^* /\!/ GL_n(\mathbb{C})$ is an algebraically completely integrable Hamiltonian system (in the Poisson sense). Roughly, their perspective is to regard this as a space of Higgs bundles rather than as meromorphic connections, but of course a trivial vector bundle over a compact Riemann surface has a canonical flat connection which gives a simple relationship between these two viewpoints.

2.2. **Symplectic Leaves.** The next step is to examine the symplectic leaves of the Poisson quotient $\mathfrak{g}^*_{k_1} \times \cdots \times \mathfrak{g}^*_{k_m} // GL_n(\mathbb{C})$. Firstly the symplectic leaves of $\mathfrak{g}^*_{k_1} \times \cdots \times \mathfrak{g}^*_{k_m}$ are all of the form $O_1 \times \cdots \times O_m$ where $O_i \subset \mathfrak{g}^*_{k_i}$ is a coadjoint orbit of G_{k_i}. It follows that the symplectic leaves of $\mathfrak{g}^*_{k_1} \times \cdots \times \mathfrak{g}^*_{k_m} // GL_n(\mathbb{C})$ are the symplectic quotients:

$$O_1 \times \cdots \times O_m // GL_n(\mathbb{C})$$

The key fact now (which is essentially well known) is:

Proposition 2.4. *Two nice meromorphic connections are formally equivalent at a_i if and only if their polar parts at a_i lie in the same coadjoint orbit in $\mathfrak{g}^*_k$.*

Not much more work is then required to prove:

Theorem 2.5. *Let $\mathbf{A} = (\ldots, {}^i\!A^0, \ldots)$ be an m-tuple of nice formal normal forms (one at each a_i) and let O_i be the G_{k_i} coadjoint orbit through ${}^i\!A^0$ (which is regarded as an element of $\mathfrak{g}^*_{k_i}$ in the way described above). Then:*

- *The set $\mathcal{P}(\mathbf{A})$ of isomorphism classes of pairs (V, ∇) where V is a trivial rank n holomorphic vector bundle over $\mathbb{P}^1$ and ∇ is a meromorphic connection on V with formal normal form ${}^i\!A^0$ at a_i for each i and no other poles is isomorphic to the symplectic quotient of $O_1 \times \ldots \times O_m$ by $GL_n(\mathbb{C})$ at the value 0 of the moment map:*

$$\mathcal{P}(\mathbf{A}) \cong O_1 \times \cdots \times O_m // GL_n(\mathbb{C}) = (\mu|_{O_1 \times \cdots \times O_m})^{-1}(0)/GL_n(\mathbb{C})$$

- *In this way $\mathcal{P}(\mathbf{A})$ inherits an intrinsic symplectic structure. That is, the symplectic structure obtained is not dependent on the coordinate choice.*

2.3. **Extended Polar Parts Manifolds.** The story for the extended polar parts manifolds is similar. The G_k coadjoint orbits are replaced by larger symplectic manifolds, 'extended orbits', encoding the compatible framings and allowing the residues of the formal normal forms to vary. To start with some more facts about the groups G_k are needed for $k \leq -2$.

There is an exact sequence of groups

$$1 \longrightarrow B_k \longrightarrow G_k \longrightarrow GL_n(\mathbb{C}) \longrightarrow 1$$

where the projection onto $GL_n(\mathbb{C})$ is evaluation at $\zeta = 0$ and so the kernel B_k is the unipotent subgroup of elements with constant term 1. The induced surjection $\pi : \mathfrak{g}^*_k \to \mathfrak{b}^*_k$ on the duals of the Lie algebras may be thought of as 'forgetting the residue'.

Let $\mathfrak{t} \subset \mathrm{End}(\mathbb{C}^n)$ be the subset of diagonal matrices and choose $A^0_k, \ldots, A^0_{-2} \in \mathfrak{t}$ such that A^0_k has distinct diagonal entries. The element

$$A^0_B := \left(A^0_k \zeta^k + \cdots + A^0_{-2}\zeta^{-2}\right) d\zeta$$

of $\mathfrak{g}^*_k$ will be regarded as an element of $\mathfrak{b}^*_k$ (the dual of the Lie algebra of B_k) via the projection π. Let O_B be the B_k-coadjoint orbit through A^0_B and observe that each element in this coadjoint orbit has the same leading term $A^0_k \zeta^k d\zeta$.

Definition 2.6. The *extension* of the B_k coadjoint orbit O_B is the set:

$$\widetilde{O}_B := \{(g_0, A) \in GL_n(\mathbb{C}) \times \mathfrak{g}_k^* \mid \pi(g_0 A g_0^{-1}) \in O_B\}$$

The element A in a pair $(g_0, A) \in \widetilde{O}_B$ should be thought of as the polar part of a connection and g_0 as a compatible framing. The condition on (g_0, A) amounts to fixing the irregular part of the formal normal form to be that specified by A_B^0: if $(g_0, A) \in \widetilde{O}_B$ then A is completely diagonalisable; for some $g \in G_k$:

$$A = g(A_k^0 \zeta^k + \cdots + A_{-2}^0 \zeta^{-2} + A_{-1}^0 \zeta^{-1}) g^{-1} d\zeta$$

for a uniquely determined $A_{-1}^0 \in \mathfrak{t}$. This procedure of taking the residue of the diagonalisation of A defines a surjective map

$$\mu_{\mathfrak{t}} : \widetilde{O}_B \longrightarrow \mathfrak{t}$$

There is a lot more structure to observe:

Lemma 2.7.

- *(Decoupling) There is a complex analytic isomorphism*

$$\widetilde{O}_B \cong O_B \times T^*GL_n(\mathbb{C}); \quad (g_0, A) \mapsto (\pi(g_0 A g_0^{-1}), (g_0, \mathrm{Res}(A)))$$

where $T^*GL_n(\mathbb{C}) \cong GL_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})^*$ via the left trivialisation. In particular $\widetilde{O}_B$ is a smooth holomorphic symplectic manifold.

- *There is a free Hamiltonian action of $GL_n(\mathbb{C})$ on $\widetilde{O}_B$; the action of $h \in GL_n(\mathbb{C})$ is:*

$$(g_0, A) \mapsto (g_0 h^{-1}, hAh^{-1})$$

The moment map is given by taking the residue of A: $(g_0, A) \mapsto \mathrm{Res}(A)$. Under the decoupling isomorphism above, $GL_n(\mathbb{C})$ acts only on $T^*GL_n(\mathbb{C})$ and so the symplectic quotient at the value 0 of the moment map is just O_B:

$$\widetilde{O}_B // GL_n(\mathbb{C}) \cong O_B$$

- *There is a free Hamiltonian action of $(\mathbb{C}^*)^n$ on $\widetilde{O}_B$; if $t \in (\mathbb{C}^*)^n$ then its action on $\widetilde{O}_B$ is:*

$$(g_0, A) \mapsto (tg_0, A)$$

The map $\mu_{\mathfrak{t}}$ defined above is a moment map for this action. The symplectic quotient at the value A_{-1}^0 of the moment map is the G_k-coadjoint orbit through the element of $\mathfrak{g}_k^*$ mapping onto $A_B^0 \in \mathfrak{b}^*$ under π and having residue A_{-1}^0:

$$\widetilde{O}_B //_{A_{-1}^0} (\mathbb{C}^*)^n \cong O(A_B^0 + A_{-1}^0 \zeta^{-1} d\zeta)$$

Sketch. The simplest way to deduce these results is to start with T^*G_k. Left multiplication defines an action of B_k on G_k which lifts to a Hamiltonian action on T^*G_k. Then observe $\widetilde{O}_B$ (with the symplectic structure specified above) is the symplectic quotient of T^*G_k by B_k over the coadjoint orbit O_B □

There are similar 'extended orbits' in the simple pole case $k_i = -1$ but they will not be discussed here.

The next step is to globalise to obtain the extended polar parts manifolds. Choose ${}^iA^0_{k_i}, \dots, {}^iA^0_{-2} \in \mathfrak{t}$ for $i = 1, \dots, m$ such that each ${}^iA^0_{k_i}$ has distinct diagonal entries. Let $O_{B_i} \subset \mathfrak{b}^*_{k_i}$ be the corresponding coadjoint orbit for each i. The main result now is

Theorem 2.8.

• *The set of isomorphism classes of triples $(V, \nabla, \mathbf{g})$ consisting of a nice meromorphic connection ∇ on trivial V having fixed irregular part of formal normal form at each a_i (but with arbitrary residue) along with an m-tuple of compatible framings $\mathbf{g} = ({}^1g, \dots, {}^mg)$ is isomorphic to the symplectic quotient of $\widetilde{O}_{B_1} \times \cdots \times \widetilde{O}_{B_m}$ $GL_n(\mathbb{C})$ at the value 0 of the moment map:*

$$\mathcal{P}_{\text{ext}}(\mathbf{A}) \cong \widetilde{O}_{B_1} \times \cdots \times \widetilde{O}_{B_m} // GL_n(\mathbb{C})$$

• *In this way $\mathcal{P}_{\text{ext}}(\mathbf{A})$ inherits an intrinsic symplectic structure.*

• *The polar parts manifold $\mathcal{P}(\mathbf{A})$ is a symplectic quotient of $\mathcal{P}_{\text{ext}}(\mathbf{A})$ by a torus $(\mathbb{C}^*)^{nm}$ whose moment map fixes the values of the residues of the formal normal forms.*

Using Lemma (2.7) it then follows that

$$\mathcal{P}_{\text{ext}}(\mathbf{A}) \cong O_{B_1} \times \cdots \times O_{B_m} \times \big(T^*(GL_n(\mathbb{C})^m) // GL_n(\mathbb{C})\big)$$

and hence

Corollary 2.9.

• *The extended polar parts manifold is a (genuine) complex manifold.*

• *The symplectomorphism type of $\mathcal{P}_{\text{ext}}(\mathbf{A})$ is independent of the irregular part of the formal normal forms (since the O_{B_i}'s are coadjoint orbits of nilpotent Lie groups and therefore admit global Darboux coordinates).*

This last fact is crucial to understanding irregular isomonodromic deformations.

3. GENERALISED MONODROMY MANIFOLDS

A meromorphic connection on $\mathbb{P}^1$ with poles at $a_1, \dots, a_m$ determines (up to conjugacy) a monodromy representation:

$$\rho : \pi_1(\mathbb{P}^1 \setminus \{a_1, \dots, a_m\}) \to GL_n(\mathbb{C}) \tag{4}$$

This gives a map, the *monodromy map*, from connections to representations of the fundamental group. If restricted to connections with only simple poles and fixed formal normal forms, this monodromy map is injective, but for higher order poles local information at each a_i is lost.

The precise data encoding the local information in the germ of a meromorphic connection at a pole has been described by a number of people (c.f. [6, 5, 19]). That is, they give an explicit parameterisation of the holomorphic equivalence classes of meromorphic connection germs. Without going into the details, this data is stored in 'Stokes matrices' which can be thought of as a generalisation of the monodromy matrices in (4).

A bit more precisely, there is a collection of 'Stokes groups' (housing the Stokes matrices) associated to each pole a_i:

$$ {}^i\mathrm{Sto}_j \subset GL_n(\mathbb{C}) \qquad j = 1, \dots, 2(|k_i| - 1) $$

The odd Stokes groups (${}^i\mathrm{Sto}_1, {}^i\mathrm{Sto}_3, \dots$) are equal to the group of lower triangular matrices with ones on the diagonal and the even Stokes groups (${}^i\mathrm{Sto}_2, {}^i\mathrm{Sto}_4, \dots$) are equal to the group of upper triangular matrices with ones on the diagonal. The classification result can be paraphrased as

Theorem 3.1. [6, 5, 19] *The set of isomorphism classes of pairs (∇, g) consisting of a meromorphic connection germ ∇ at a_i (formally equivalent to ${}^iA^0$) and a compatible framing g, is naturally isomorphic to the product of the Stokes groups at a_i:*

$$ {}^i\mathrm{Sto}_1 \times \cdots \times {}^i\mathrm{Sto}_{2(|k_i|-1)} $$

In particular it is isomorphic to an even dimensional complex vector space.

The diagonal subgroup $T = (\mathbb{C}^*)^n \subset GL_n(\mathbb{C})$ acts by conjugation on the Stokes groups and this action corresponds to changing the choice of compatible framing:

Corollary 3.2. [6, 5, 19] *The quotient ${}^i\mathrm{Sto}_1 \times \cdots \times {}^i\mathrm{Sto}_{2(|k_i|-1)}/T$ is isomorphic to the set of isomorphism classes of meromorphic connection germs at a_i (that are all formally equivalent to ${}^iA^0$).*

Global meromorphic connections with compatible framings are then classified by their Stokes matrices at each a_i together with 'connection matrices' $C_2, \dots, C_m \in GL_n(\mathbb{C})$ describing the relation between the framings at a_1 and each of the other poles a_i in turn.

This data (the Stokes and connection matrices) is subject to one condition, ensuring that the monodromy around a contractible loop is the identity:

$$ (C_m^{-1} \cdot {}^m\Lambda \cdot C_m) \cdots (C_3^{-1} \cdot {}^3\Lambda \cdot C_3)(C_2^{-1} \cdot {}^2\Lambda \cdot C_2) \cdot {}^1\Lambda = 1 \tag{5} $$

with

$$ {}^i\Lambda := {}^iS_{2(|k_i|-1)} \cdots {}^iS_2 {}^iS_1 {}^iM_0 \text{ for } i = 1, \dots, m $$

where ${}^iS_j \in {}^i\mathrm{Sto}_j$ is a Stokes matrix, and ${}^iM_0 := \exp(2\pi i({}^iA^0{}_{-1}))$ is the 'formal monodromy'. This leads us to define the affine variety:

$$ \widetilde{M}(\mathbf{A}) := \left\{ ({}^1\mathbf{S}, \dots, {}^m\mathbf{S}, \mathbf{C}) \in {}^1\mathbf{Sto} \times \cdots \times {}^m\mathbf{Sto} \times GL_n(\mathbb{C})^{m-1} \,\middle|\, \text{relation (5) holds} \right\} $$

where bold typeface denotes the appropriate tuples. The m tori (T's) acting on the Stokes groups at each a_i act on the connection matrices as well. This gives an action of $T^m \cong (\mathbb{C}^*)^{nm}$ on $\widetilde{M}(\mathbf{A})$ given explicitly as:

$$ \mathbf{t}({}^iS_j) = t_i {}^iS_j t_i^{-1} \quad \text{for } j = 1, \dots, 2(|k_i| - 1),\ i = 1, \dots, m, \text{ and} $$
$$ \mathbf{t}(C_i) = t_i C_i t_1^{-1} \quad \text{for } i = 2, \dots, m. $$

where $\mathbf{t} = (t_1, \dots, t_m) \in T^m \cong (\mathbb{C}^*)^{nm}$.

Definition 3.3.

• The 'monodromy manifold' $M(\mathbf{A})$ is the quotient $\widetilde{M}(\mathbf{A})/(\mathbb{C}^*)^{nm}$.

• The 'extended monodromy manifold' $M_{\text{ext}}(\mathbf{A})$ is the disjoint union of the $\widetilde{M}(\mathbf{A})$'s over all the possible choices of the residues of the formal normal forms $\mathbf{A}$.

The main result of this section is

Theorem 3.4.

• *The monodromy manifold $M(\mathbf{A})$ is isomorphic to the set of isomorphism classes of pairs (V, ∇) where V is a degree zero rank n holomorphic vector bundle over $\mathbb{P}^1$ and ∇ is a meromorphic connection on V with formal normal form ${}^iA^0$ at a_i for each i and no other poles.*

• *The extended monodromy manifold $M_{\text{ext}}(\mathbf{A})$ is isomorphic to the set of isomorphism classes of triples $(V, \nabla, \mathbf{g})$ consisting of a nice meromorphic connection ∇ on degree zero V having fixed irregular part of formal normal form at each a_i (but with arbitrary residue of formal normal form) along with an m-tuple of compatible framings.*

• *$M_{\text{ext}}(\mathbf{A})$ is a complex manifold and has $M(\mathbf{A})$ as the subquotient obtained by fixing the residues of the formal normal forms and quotienting by T^m.*

Remark 3.5.

• The monodromy manifolds $M(\mathbf{A})$ are regarded as natural generalisation of the symplectic leaves of the moduli space of flat $GL_n(\mathbb{C})$ connections on $\mathbb{P}^1 \setminus \{a_i\}$ since in the simple pole case that is what they are.

• In particular $M(\mathbf{A})$ should have a complex symplectic structure. A finite dimensional construction of such is not known. This would generalise the host of results in the simple pole case. A symplectic structure is provided formally in the next section by describing $M(\mathbf{A})$ as an infinite dimensional symplectic quotient.

Example 3.6. Consider the case when there are only two marked points: $a_1 = 0, a_2 = \infty$ and the formal normal form at each a_i has a pole of order 2:

$$ {}^1A^0 = A\frac{dz}{z^2} + Q\frac{dz}{z}, \qquad\qquad {}^2A^0 = -Bdz - R\frac{dz}{z} $$

where each $n \times n$ matrix A, B, Q, R is diagonal and A, B have distinct eigenvalues. Isomonodromic deformations of connections with such formal normal forms yield the third Painlevé equation in the 2×2 case. There is one connection matrix $C \in GL_n(\mathbb{C})$ and at each a_i there are two Stokes matrices ${}^iL, {}^iU$ one lower triangular and one upper triangular. The extended monodromy manifold is simply

$$ M_{\text{ext}}(\mathbf{A}) := \{{}^1L, {}^2L, {}^1U, {}^2U, C, Q, R \mid C({}^1U \cdot {}^1L \cdot e^Q)C^{-1}({}^2U \cdot {}^2L \cdot e^R) = 1\} $$

One may see this is a nonsingular manifold by observing it is a covering of its image under the obvious projection to the manifold $\{{}^1L, {}^1U, C, Q\}$. This image is open.

4. C^∞ APPROACH TO MEROMORPHIC CONNECTIONS

Recall that D is the effective divisor $-k_1(a_1) - \cdots - k_m(a_m)$ on $\mathbb{P}^1$. Define the sheaf of 'smooth functions with poles on D' to be

$$C^\infty_D = \mathcal{O}[D] \otimes_{\mathcal{O}} C^\infty$$

where $\mathcal{O}$ is the sheaf of holomorphic functions and C^∞ the infinitely differentiable complex functions. Any local section of C^∞_D near a_i is of the form $f(z)(z-a_i)^{k_i}$ for a C^∞ function f. Similarly define sheaves $\Omega^{p,q}_D$ of C^∞ (p,q)-forms with poles on D (so in particular $C^\infty_D = \Omega^{(0,0)}_D$). A basic feature is that 'smooth Laurent expansions' can be taken at each a_i. This gives a map

$$L_i : \Omega^*_D(\mathbb{P}^1) \to \mathbb{C}[\![z_i, \bar{z}_i]\!] z_i^{k_i} \otimes \Lambda^* \mathbb{C}^2$$

where $z_i := (z - a_i)$ and $\mathbb{C}^2 = \mathbb{C}dz_i \oplus \mathbb{C}d\bar{z}_i$. For example $L_i(f(z)(z-a_i)^{k_i}) = T_i(f)z_i^{k_i}$ where $T_i(f)$ is the Taylor expansion of the C^∞ function f at a_i.

The Laurent map L_i has nice morphism properties: $L_i(\omega_1 \wedge \omega_2) = L_i(\omega_1) \wedge L_i(\omega_2)$ (provided the product doesn't have poles of too high order) and L_i commutes with the exterior derivative d, where d is defined on the right-hand side in the obvious way (for example $d(z_i^{-1}) = -dz_i/z_i^2$).

Now let $E \to \mathbb{P}^1$ be a trivial C^∞ rank n complex vector bundle and choose a trivialisation of it (so sections of E are identified with column vectors of functions etc). Define a space of C^∞ connections on E with poles (on D) in their (1,0) parts:

$$\mathcal{A} = \left\{ d + A \middle| A \in \Omega^{1,0}_D(\mathbb{P}^1, \mathrm{End}(E)) \oplus \Omega^{0,1}(\mathbb{P}^1, \mathrm{End}(E)) \right\}$$

This is acted on by the gauge group $\mathcal{G} = \mathrm{Aut}(E) \cong GL_n(C^\infty(\mathbb{P}^1))$ (using the trivialisation). Recall that at each a_i a nice diagonal formal normal form ${}^iA^0 = {}^iA^0{}_{k_i}(z-a_i)^{k_i}dz + \cdots + {}^iA^0{}_{-1}dz/(z-a_i)$ has been chosen and that $\mathbf{A}$ denotes the m-tuple $(\ldots, {}^iA^0, \ldots)$ of formal normal forms.

Definition 4.1.

- The space of singular connections with fixed Laurent expansions is

$$\mathcal{A}(\mathbf{A}) := \left\{ d + A \in \mathcal{A} \;\middle|\; L_i(A) = {}^iA^0 \text{ for each } i \right\}$$

- The group of gauge transformations whose Taylor series preserve the formal normal forms is

$$\mathcal{G}(\mathbf{A}) := \left\{ g \in \mathcal{G} \;\middle|\; (L_i(g))[{}^iA^0] = {}^iA^0 \text{ for each } i \right\}$$

where the square brackets [ ] denote the gauge action.

- The 'curvature' of $d + A \in \mathcal{A}(\mathbf{A})$ is the nonsingular matrix of two-forms $dA + A^2$.
- The 'flat' connections are those with zero curvature.

Observe $\mathcal{A}(\mathbf{A})$ is an affine space modelled on the C^∞ matrices of one-forms having zero Taylor expansion at each marked point a_i. Also $\mathcal{G}(\mathbf{A})$ turns out to be the set of bundle automorphisms $g \in \mathcal{G}$ which have Taylor expansion equal to a constant diagonal matrix at each a_i. The main observation now is:

Proposition 4.2. *There is a natural (set-theoretical) bijection between the quotient of the flat connections with fixed Laurent expansions by the group of bundle automorphisms whose Taylor series preserve the formal normal forms, and the set of isomorphism classes of pairs (V, ∇) where V is a degree zero holomorphic vector bundle over $\mathbb{P}^1$ and ∇ is a meromorphic connection on V with poles only at the marked points a_i and is formally equivalent to the formal normal form ${}^iA^0$ at a_i for each $i = 1, \dots, m$:*

$$\mathcal{A}_{\text{flat}}(\mathbf{A})/\mathcal{G}(\mathbf{A}) \cong M(\mathbf{A})$$

Sketch. Recall $M(\mathbf{A})$ describes meromorphic connections on degree zero bundles. Given $d + A \in \mathcal{A}_{\text{fl}}(\mathbf{A})$, set $\nabla = d + A$ and let V be E with holomorphic structure determined by the $(0,1)$-part of ∇. Flatness implies ∇ is meromorphic on V. Now the fact that A has Laurent expansion ${}^iA^0$ at a_i translates into the fact that ∇ is formally equivalent to ${}^iA^0$ (a formal transformation between the germ of ∇ at a_i and the formal normal form ${}^iA^0$ is given by the Taylor series of the gauge transformation from the fixed trivialisation to a local holomorphic trivialisation of V) □

Remark 4.3. The extended monodromy manifold $M_{\text{ext}}(\mathbf{A})$ may be obtained similarly: a larger affine space $\mathcal{A}_{\text{ext}}(\mathbf{A})$ is defined in which the residues of the formal normal forms are arbitrary diagonal matrices and $\mathcal{G}(\mathbf{A})$ is replaced by the group $\mathcal{G}_1$ of bundle automorphisms with Taylor expansion 1 at each a_i. Observe $\mathcal{G}(\mathbf{A})/\mathcal{G}_1 \cong (\mathbb{C}^*)^{nm}$.

4.1. **Symplectic Nature.** The next step is to interpret the quotient $\mathcal{A}_{\text{flat}}(\mathbf{A})/\mathcal{G}(\mathbf{A})$ as an infinite dimensional complex symplectic quotient. This is only done in a formal sense here: appropriate Banach space methods are not known (at least to the author). It is believed that this is just a technical difficulty and that the procedure here does define a symplectic form on the extended monodromy manifold $M_{\text{ext}}(\mathbf{A})$ (and on the smooth part of $M(\mathbf{A})$). Some justification for this comes from two facts: firstly in the case when all the poles are simple, our procedure agrees with the much studied symplectic structure on the moduli space of flat connections on the punctured $\mathbb{P}^1$ with fixed monodromy conjugacy classes. Secondly Theorem 4.5 below implies that a symplectic form is obtained in this way on a dense open subset of the monodromy manifolds (namely the image of the polar parts manifolds under the monodromy map).

The main result (the symplectic description of the IMD equations) is independent of this inadequacy though. The symplectic nature of $M(\mathbf{A})$ serves more to explain *why* the IMD equations have a symplectic description. This surely merits further study and a finite dimensional construction of the symplectic structure should be possible, maybe building on those of [11, 2, 10] in the nonsingular case or [8, 20] in the parallel world of meromorphic Higgs bundles.

The C^∞ picture is based on that of Atiyah-Bott [3] for nonsingular unitary connections over arbitrary compact Riemann surfaces (the lectures [4] of M.Audin give an overview of the symplectic geometry involved):

Let $\alpha \in \mathcal{A}(\mathbf{A})$ be a singular connection with fixed Laurent expansions. The expression

$$\omega_\alpha(\phi, \psi) = \int_{\mathbb{P}^1} \mathrm{Tr}(\phi \wedge \psi)$$

defines a two-form on the tangent space $T_\alpha \mathcal{A}(\mathbf{A})$. This determines a closed differential two-form ω on the affine space $\mathcal{A}(\mathbf{A})$. It is nondegenerate in the sense that if $\omega_\alpha(\phi, \psi) = 0$ for all ψ then $\phi = 0$.

Proposition 4.4.

- *The gauge action of $\mathcal{G}(\mathbf{A})$ on $\mathcal{A}(\mathbf{A})$ preserves the symplectic form ω.*
- *Formally, the curvature is a moment map for this action.*

Thus if this formal picture was made rigorous, the symplectic quotient at the value 0 of the moment map is just the subset of flat connections modulo the gauge group and so via the isomorphism in Proposition 4.2, the monodromy manifold $M(\mathbf{A})$ inherits a symplectic structure.

The symplectic form ω on $\mathcal{A}(\mathbf{A})$ can be calculated though and essentially Stokes' theorem yields:

Theorem 4.5. *The monodromy map $\nu : \mathcal{P}_{\mathrm{ext}}(\mathbf{A}) \to M_{\mathrm{ext}}(\mathbf{A})$ is symplectic; more precisely, locally on $\mathcal{P}_{\mathrm{ext}}(\mathbf{A})$, ν lifts to a map $\mathcal{P}_{\mathrm{ext}}(\mathbf{A}) \to \mathcal{A}_{\mathrm{ext,flat}}(\mathbf{A}) \subset \mathcal{A}_{\mathrm{ext}}(\mathbf{A})$ and this is symplectic.*

5. ISOMONODROMIC DEFORMATIONS

The final step is to examine how the picture outlined above behaves as the irregular parts of the formal normal forms $\mathbf{A}$ and the positions $a_1, \dots, a_m$ of the poles are allowed to vary. The positive integers n and m and negative integers $k_1, \dots, k_m$ remain fixed throughout.

Definition 5.1. The space X of deformation parameters is

$$X := X_{\mathrm{mp}} \times X_1 \times \cdots \times X_m$$

where X_{mp} parameterises the marked points $a_1, \dots, a_m$:

$$X_{\mathrm{mp}} := (\mathbb{P}^1)^m \setminus \text{diagonals}.$$

and X_i parameterises the irregular parts of nice degree k_i formal normal forms: $X_i = \{0\}$ if $k_i = -1$ and if $k_i \leq -2$:

$$X_i := \left\{ ({}^iA_{k_i}, \dots, {}^iA_{-2}) \in (\mathfrak{t} \setminus \Delta) \times \mathfrak{t}^{|k_i|-2} \right\}$$

where $\mathfrak{t} \cong \mathbb{C}^n$ is the set of diagonal matrices in $\mathrm{End}(\mathbb{C}^n)$ and $\Delta \subset \mathbb{C}^n$ is the set of diagonals.

Observe X_i is homotopy equivalent to $\mathbb{C}^n \setminus \Delta$ and recall that the fundamental group of this is a braid group.

Each point $x \in X$ determines an m-tuple $\mathbf{A}(x)$ of irregular parts of formal normal forms and the positions of m distinct points $a_1, \dots, a_m \in \mathbb{P}^1$ and so determines an extended polar parts manifold $\mathcal{P}_x := \mathcal{P}_{\text{ext}}(\mathbf{A}(x))$ and an extended monodromy manifold $M_x := M(\mathbf{A}(x))$. These fit together into bundles over the space of deformation parameters:

Proposition 5.2. *There is a locally trivial fibre bundle $\mathcal{P} \to X$ (resp. $M \to X$), the polar parts (resp. monodromy) bundle, having $\mathcal{P}_x$ (resp. M_x) as fibre over $x \in X$.*

In fact, from their description in terms of Stokes and connection matrices it can be observed that over a contractible patch $U \subset X$ all of the monodromy manifolds M_x are isomorphic. Infinitesimally this identification gives:

Proposition 5.3. *There is a flat (Ehresmann) connection on the mondromy bundle M transverse to the fibres of the projection onto X.*

Remark 5.4. Globally this identification cannot be made since local (on X) choices are made to relate the monodromy manifolds (thought of abstactly in terms of meromorphic connections) to the Stokes and connection matrices. This fact leads to braid group actions on the monodromy manifolds.

This connection identifies the (generalised) monodromy data of meromorphic connections on $\mathbb{P}^1$ which have different formal normal forms/pole positions and so will be called[2] the *isomonodromy* connection. Under this identification points on the same horizontal leaf of the isomonodromy connection have the *same* monodromy.

The monodromy map extends to a bundle map $\nu : \mathcal{P} \to M$ and the isomonodromy connection can be pulled back to $\mathcal{P}$ (this will be called the isomonodromy connection on $\mathcal{P}$).

The isomonodromic deformation equations introduced by Jimbo-Miwa-Ueno [18] are then succinctly described as being this connection on the polar parts bundle $\mathcal{P}$. See Figure 1. Observe immediately that from this perspective

- The question of Frobenius integrability is transparent- the integral leaves are manifest.
- The flows occur (essentially) on a family of coadjoint orbits. Even this appears to be new in the general case.
- Since the image $\nu(\mathcal{P})$ is only a subset of M some flows in $\mathcal{P}$ will not be complete; solutions will have singularities. The 'Painlevé Property' of the equations says that these singularities will be at worst poles.

The main new result here is:

Theorem 5.5.
The isomonodromy connection on the bundle $\mathcal{P}$ of polar parts manifolds is symplectic.

[2] A better name might be the 'Gauss-Manin connection' , by analogy with that in non-Abelian cohomology [23].

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